Sains Malaysiana 48(12)(2019): 2787–2796
http://dx.doi.org/10.17576/jsm-2019-4812-20

A Distance Measure of Interval-valued Belief Structures

(Suatu Jarak Pengukuran Nilai Selang Struktur Kepercayaan)

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ABSTRACT

Interval-valued belief structures are generalized from belief function theory, in terms of basic belief assignments from crisp to interval numbers. The distance measure has long been an essential tool in belief function theory, such as conflict evidence combinations, clustering analysis, belief function and approximation. Researchers have paid much attention and proposed many kinds of distance measures. However, few works have addressed distance measures of interval-valued belief structures up. In this paper, we propose a method to measure the distance of interval belief functions. The method is based on an interval-valued one-dimensional Hausdorff distance and Jaccard similarity coefficient. We show and prove its properties of non-negativity, non-degeneracy, symmetry and triangle inequality. Numerical examples illustrate the validity of the proposed distance.

Keywords: Distance; Hausdorff distance; interval-valued belief structures; Jaccard similarity coefficient

ABSTRAK

Nilai selang struktur kepercayaan digeneralisasi daripada teori fungsi kepercayaan, dari sudut tugasan kepercayaan asas nombor krisp kepada selang. Jarak pengukuran telah menjadi alat yang penting dalam teori fungsi kepercayaan, seperti gabungan bukti konflik, analisis berkelompok, fungsi kepercayaan dan penganggaran. Penyelidik telah memberi banyak perhatian dan mencadangkan pelbagai jenis jarak pengukuran. Walau bagaimanapun, beberapa kajian telah membincangkan jarak pengukuran nilai selang struktur kepercayaan. Dalam kertas ini, kami mencadangkan kaedah untuk mengukur jarak fungsi selang kepercayaan. Kaedah ini berdasarkan jarak nilai selang satu dimensi Hausdorff dan pekali kesamaan Jaccard. Kami tunjuk dan buktikan sifatnya yang tidak negatif, tidak merosot, simetri dan ketidaksamaan segitiga. Contoh berangka menunjukkan kesahan jarak yang dicadangkan.

Kata kunci: Jarak; jarak Hausdorff; nilai selang struktur kepercayaan; pekali kesamaan Jaccard

INTRODUCTION

The Dempster-Shafer theory (DST), also called evidence theory or belief function theory, was presented by Dempster (1967), and extended and refined by his student, Glenn Shafer (1976). It has been developed and applied in areas such as fault diagnosis (Wu et al. 2010; Yuan et al. 2016), decision making (Bauer 1997; Giang 2015), pattern classification (Liu et al. 2014; Thierry & Philippe 2007), and clustering (Hariz et al. 2006). Among the tools developed to work with DST, distances have recently received increased attention and much work on measuring the distance or dissimilarity between two belief functions has emerged (Jousselme & Maupin 2012). The distance measure can describe the degree of dissimilarity or similarity between bodies of evidence (BOE), and that has been proposed as a tool in various applications including conflict evidence combination (Deng et al. 2004; Martin et al. 2008), clustering analysis (Deneux 2000), learning models (Zouhal 1998) and belief function approximation (Cuzzolin 2011; Klein et al. 2016; Tessem 1993).

However, in practice, incompleteness or lack of information causes partial or total ignorance, and assigning a crisp number to every focal element is often regarded as too restrictive. Interval-valued data arise in situations requiring management of either the uncertainty related to measurements or the variability inherent in a group rather than an individual. In the literature, Denoeux (1999), Lee and Zhu (1992), and Yager (2001) have attempted to extend DST to interval-valued belief structures (IBS). Within the framework of the transferable belief model (TBM), Denoeux extended the main concepts of DST, which include credibility, plausibility, combination and normalization which lay the theoretical foundations of IBS. Most research fields of IBS involve combination rule (Fu & Yang 2012, 2011; Sevastianov 2012; Song et al. 2014; Wang 2007), normalization (Sevastjanov et al. 2010; Xu et al. 2012), and uncertainty measure (Jiang 2017; Son 2016). However, few are concerned with the distance within the framework of IBS.

Among the referenced papers, the distances of DST can be roughly divided into direct and indirect distances (Loudahi et al. 2016). A type of direct distance, Jousselme's distance (Jousselme et al. 2001) is essentially a weighted Euclidean distance using the Jaccard similarity coefficient to measure the similarity of focal elements. It has proven highly attractive because it satisfies the mathematical constraints of a metric distance (non-negativity, non-degeneracy, symmetry and triangle inequality). A variety of other distances have since been proposed using different similarity functions between focal elements (Diaz et al. 2006; Sunberg et al. 2013). Sunberg used Hausdorff-based measure to account for the distance between focal elements which must be orderable sets. Mo et al. (2016) proposed a generalized method to measure the evidence distance which combines Jousselme's distance and Sunberg's distance by tunable parameters. Among indirect distances, the pignistic probability transform turns a belief function into the least committed probability distribution. Tessem's distance (also called the betting commitment distance or the pignistic probability distance) is used to compute approximations of basic belief assignments. Another kind of indirect distance measure is based on belief intervals (Han et al. 2014; Yang & Han 2015) which transform the evidential distance to the distance of intervals. For a thorough survey of evidential distances and their classification, see Jousselme et al. (2001).

In view of the importance of interval-valued evidential distances, we propose a distance in IBS. We use the Jaccard similarity coefficient to measure the similarity of focal elements and the Hausdorff distance to measure the distance between interval numbers. The proposed interval distance can measure both the crisp and the interval-valued belief structures (Fallatah et al. 2018; Huh et al. 2018; Kaushik & Chatterjee 2018; Li et al. 2018; Yang et al. 2018).

The rest of this paper is organized as follows. Next, we briefly review the fundamental notions of evidence theory and interval-valued belief structures (Oyekale 2017; Pedroza et al. 2017; Sigren 2018; Skibicki 2017; Wahi et al. 2018). Some evidential distances in DST literature are discussed in subsequent section. A new distance of interval evidence is proposed in the following section. Some desired properties and related proofs about the distance are also provided. Experiments and simulations are described in the last section.

BACKGROUND

We review some basic concepts commonly used in DST and IBS.

BASICS OF BELIEF FUNCTIONS

The basic concepts of DST were first introduced by Dempster and developed by Shafer. The following definitions are central in evidence theory.

Definition 1 Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be a finite set of mutually exclusive sets of propositions, referred to as a frame of discernment (FOD). A basic belief assignment (BBA) m, is defined as a mapping from 2^Θ to 1 (m: $2^\Theta \rightarrow [0, 1]$), which satisfies,

$$\sum_{A\subseteq\Theta} m(A) = 1, m(\emptyset) = 0. \quad (1)$$

A set A is a focal element of m iff $m(A) > 0$. A BBA is also called a mass function. The set of all focal elements and their corresponding mass assignments constitutes a body of evidence (BOE).

Definition 2 The belief and plausibility commonality functions of a set A are respectively defined as,

$$Bel(A) = \sum_{B\subseteq A} m(B), \quad (2)$$

$$Pl(A) = \sum_{A\cap B\neq\emptyset} m(B). \quad (3)$$

$Bel(A)$ represents how much the event A is implied by the actual evidence, i.e. how much the justified specific support exits for focal element A. $Pl(A)$ represents how consistent the event A is with the actual evidence, i.e. how much the potential specific support exists for A. The length of the belief interval $[Bel(A), Pl(A)]$ represents the degree of imprecision for A.

Definition 3 The pignistic probability corresponding to a BBA is defined by,

$$BetP_m(A) = \sum_{B\in 2^\Theta} \frac{|A\cap B|}{|B|} m(B). \quad (4)$$

Called the *betting commitment to A*. $BetP_m(A)$ provides the total mass value that A can carry.

The core of evidence theory is Dempster's rule of combination by which evidence from different sources is combined.

Definition 4 With two belief structures m_1 and m_2, Dempster's rule of combination is defined as,

$$[m_1 \oplus m_2](C) = \begin{cases} \dfrac{\sum_{A\cap B=C} m_1(A)m_2(B)}{1-K} & \forall C\subseteq\Theta, C\neq\emptyset, \\ 0 & C=\emptyset \end{cases} \quad (5)$$

where $\oplus$ represents the combination operator, also called the orthogonal sum; A and B are focal elements; $K = \sum_{A\cap B=\emptyset} m_1(A)m_2(B)$ is the conflict coefficient, which measures the conflict between the pieces of evidence; and division by (1-K) is called normalization.

BASICS OF INTERVAL-VALUED BELIEF FUNCTIONS

In an IBS, belief masses are no longer described by precise numbers, but lie within certain intervals. It is constrained as follows (Smets & Kennes 1994).

Definition 5 Let Θ be the frame of discernment, $F_1, F_2, \ldots, F_N$ be N subsets of Θ and $[a_i, b_i]$ be N intervals with $0 \leq a_i \leq b_i \leq 1$ (i=1,2,…,N). An interval-valued belief structure (IBS) is a belief structure on Θ such that,

(1) $a_i \le m(F_i) \le b_i$, where $0 \le a_i \le b_i \le 1$ for i=1,2,…,N;

(2) $\sum_{i=1}^{N} a_i \le 1$ and $\sum_{i=1}^{N} b_i \ge 1$;

(3) $m(H)=0$, $\forall H \notin \{F_1, F_2, \ldots, F_N\}$.

If an IBS satisfies the above, we can say it is valid.

Definition 6 Let m be a valid IBS such that $a_i \le m(F_i) \le b_i$. If a_i and b_i satisfy

$$\sum_{j=1}^{N} b_j - (b_i - a_i) \ge 1 \text{ and } \sum_{j=1}^{N} a_j + (b_i - a_i) \le 1,\ i=1,2,\ldots,N$$

then m is said to be normalized.

A normalized IBS is valid, but the converse is not always true.

If an IBS is valid, but not normalized, then it can be normalized by (6) and (7).

$$\hat{a}_i = \frac{a_i}{a_i + \sum_{j=1, j\ne i}^{N} b_j},\ i = 1, 2, \ldots, N \tag{6}$$

$$\hat{b}_i = \frac{b_i}{b_i + \sum_{j=1, j\ne i}^{N} a_j},\ i = 1, 2, \ldots, N\ . \tag{7}$$

Any IBSs in this paper is assumed to be valid and normalized.

Definition 7 Let m_1 and m_2 be two interval-valued belief structures with interval-valued probability masses $m_1^-(A_i) \le m_1(A_i) \le m_1^+(A_i)$ for i=1 to n_1 and $m_1^-(B_j) \le m_1(B_j) \le m_1^+(B_j)$ for j=1 to n_2, respectively. Their combination, denoted by $m_1 \oplus m_2$, is also an interval-valued belief structure defined by,

$$[m_1 \oplus m_2](C) = \begin{cases} \left[(m_1 \oplus m_2)^-(C), (m_1 \oplus m_2)^+(C) \right] & \forall C \subseteq \Theta, C \ne \varnothing, \\ 0 & C = \varnothing \end{cases} \tag{8}$$

where $(m_1 \oplus m_2)^-(C)$ and $(m_1 \oplus m_2)^+(C)$ are respectively the minimum and maximum of the following pair of optimization problems:

$$\begin{aligned} \text{Max/Min} \quad & [m_1 \oplus m_2](C) = \frac{\sum_{A_i \cap B_j = C} m_1(A_i) m_2(B_j)}{1 - \sum_{A_i \cap B_j = \varnothing} m_1(A_i) m_2(B_j)} \\ \text{s.t.} \quad & \sum_{i=1}^{n_1} m_1(A_i) = 1 \\ & \sum_{j=1}^{n_2} m_2(B_j) = 1 \\ & m_1^-(A_i) \le m_1(A_i) \le m_1^+(A_i),\ i=1,2,\ldots,n_1, \\ & m_2^-(B_j) \le m_2(B_j) \le m_2^+(B_j),\ j=1,2,\ldots,n_2. \end{aligned} \tag{9}$$

Wang's combination rule is a quadratic optimization with constraint conditions and the combination and normalization of two pieces of interval evidence produces occur at the same time rather than separately.

DISTANCE IN THE THEORY OF BELIEF FUNCTIONS

A distance or dissimilarity between two BBAs can represent the degree of dissimilarity between BOEs. Much work on measuring the distance has emerged recently, but there is no distance measure for IBSs. The following common distances in DST can be roughly classified as either direct or indirect distances.

In view of the geometric interpretation (Cuzzolin 2008), basic belief assignments can be seen as vectors belonging to the simplex of a vector space E, which spanned by the elements of the power set 2^Θ and has the dimension $|2^\Theta|$. For two belief functions, the direct distance is defined directly on the space $E \times E$.

Jousselme's distance is a type of direct distance. Let m_1 and m_2 be two BBAs defined on the same FOD Θ, containing n mutually exclusive and exhaustive hypotheses. A and B are any focal elements of BBAs m_1 and m_2. Jousselme's distance, denoted by d_J, is given by,

$$d_J(m_1, m_2) = \sqrt{\frac{1}{2} (\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2)^T \mathbf{Jac} (\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2)}, \tag{10}$$

where **Jac** is a matrix whose elements are Jaccard indices for any pair of subsets of Θ

$$Jac(A_i, A_j) = \frac{|A_i \cap A_j|}{|A_i \cup A_j|}, \text{ for } A_i, A_j \in 2^\Theta \backslash \varnothing,\ i,j=1,2,\ldots,2^n-1, \tag{11}$$

where $|A|$ is the cardinality of A. Bouchard et al. (2008) gave a proof for the positive definiteness of the Jaccard index matrix that guarantees that Jousselme's distance is a full metric. The proof is based on the decomposition of the matrix into an infinite sum of positive semidefinite matrices. In fact, Jousselme's distance is an L_2 Euclidean distance with weighting matrix **Jac**.

However, when the frame of discernment Θ is orderable sets, the Jaccard index which uses the cardinality of unions and intersections between focal elements is not suitable for judging similarity. Sunberg et al. (2013) proposed a Hausdorff-based measure to account for the distance between focal elements. The distance maintains the quadratic form structure of Jousselme but replaces the similarity function with,

$$d_{\text{Haus}}(m_1, m_2) = \sqrt{\frac{1}{2} (\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2)^T \mathbf{S_{Haus}} (\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2)}, \tag{12}$$

where $\mathbf{S_{Haus}}$ is a similarity matrix. Each corresponding element is,

$$S_{Haus} = \frac{1}{1 + KH\left(A_i, A_j\right)}, \tag{13}$$

where $H(A_i, A_j)$ is the Hausdorff distance between focal elements A_i and A_j. $K > 0$ is a user-defined tuning parameter that adjusts metric responses with respect to the orderable space discretization. In this paper, K is set to 1. The Hausdorff distance between focal elements may be defined as,

$$H\left(A_i, A_j\right) = \max\left\{\left|\min\left(A_i\right) - \min\left(A_j\right)\right|, \left|\max\left(A_i\right) - \max\left(A_j\right)\right|\right\}, \tag{14}$$

Because uses the Hausdorff distance, the metric does not reach a saturated value when the two BBAs have no overlap. Moreover, it is a metric distance.

Mo et al. (2016) proposed a generalized evidence distance that combines Jousselme's distance and Sunberg's distance,

$$d_{\mathrm{G}}\left(m_1, m_2\right) = \sqrt{\frac{1}{2}\left(\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2\right)^{\mathrm{T}} \mathbf{D}\left(\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2\right)}, \tag{15}$$

where $\mathbf{D} = \alpha\mathbf{Jac} + \beta\mathbf{S}_{\mathbf{Haus}}$.The parameters α and β are user-defined weighted normalized coefficients, $\alpha, \beta \in [0,1]$, $\alpha+\beta=1$. When BBAs are un-orderable sets, α should be given a bigger weight than β. When BBAs are orderable sets, the parameter β should be given a bigger weight than α. For simplicity, both α and β are set to be 0.5. However, the metric properties are not provided.

The indirect distance is computed in a new representation space $F \times F$. The space F is generated by some new vectors that are transformed from BBAs, such as pignistic transforms and belief interval transforms.

Tessem's distance is also called the betting commitment distance or the pignistic probability distance. It is computed by,

$$d_{\mathrm{T}}\left(m_1, m_2\right) = \max_{A \subseteq \Theta}\left\{\left|BetP_1\left(A\right) - BetP_2\left(A\right)\right|\right\}, \tag{16}$$

where $BetP_1$ and $BetP_2$ are the pignistic probabilities of m_1 and m_2, respectively, according to (4). Tessem's distance belongs to the L_∞ family of Chebyshev distance and it is actually not a strict distance metric.

Han et al. (2014) proposed two distances based on the belief intervals [$Bel(A)$, $Pl(A)$], here we only make mention of the Euclidean-family belief interval-based distance denoted by d_{BI}. For a BBA, the belief intervals of different focal elements are first calculated, and these can be considered as interval numbers. Given two interval numbers $[a_1, b_1]$ and $[a_2, b_2]$, a strict distance (Wasserstein-based distance) (Verde & Irpino 2008) is defined as,

$$d^I\left([a_1,b_1],[a_2,b_2]\right) = \sqrt{\left(\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}\right)^2 + \frac{1}{3}\left(\frac{b_1-a_1}{2} - \frac{b_2-a_2}{2}\right)^2}. \tag{17}$$

Therefore, the distance between belief intervals $BI_1(A_i)$ (calculated from m_1) and $BI_2(A_i)$ (calculated from m_2) can be measured according to . Thus d_{BI} can be defined as

$$d_{\mathrm{BI}}\left(m_1, m_2\right) = \sqrt{N_c \sum_{i=1}^{2^n-1}\left[d^I\left(BI_1\left(A_i\right), BI_2\left(A_i\right)\right)\right]^2} = \sqrt{N_c \cdot \mathbf{d_I} \cdot \mathbf{d_I^T}}, \tag{18}$$

where $N_c = 1/2^{n-1}$ is the normalization factor, $i=1,2,\ldots, 2^n-1$, and the superscript T denotes the transpose. $\mathrm{d_I}$ is a $(1\times 2^n-1)$-dimensional row vector is given by,

$$\mathbf{d_I} = \left[d^I\left(BI_1\left(A_1\right), BI_2\left(A_1\right)\right), \; \cdots, d^I\left(BI_1\left(A_{2^n-1}\right), BI_2\left(A_{2^n-1}\right)\right)\right]. \tag{19}$$

Han et al. (2014) also proved that the belief interval-based distance is a strict metric distance.

The conflict coefficient defined by Dempster in (5) is probably the first quantification of the interaction between two belief functions. Hereafter denoted by d_{C} is redefine as,

$$d_{\mathrm{C}}\left(m_1, m_2\right) = \sum_{A_i \cap A_j = \varnothing} m_1\left(A_i\right) m_2\left(A_j\right). \tag{20}$$

However, in some cases it cannot truly reflect the degree of dissimilarity between two BBAs (Liu 2006). Furthermore, d_{C} is not a metric distance.

For a thorough survey of evidential distances and their classification and properties, see Jousselme and Maupin (2012).

THE PROPOSED DISTANCE OF INTERVAL-VALUED BELIEF STRUCTURE AND ITS PROPERTIES

We now present the interval distance between two IBSs based on the Hausdorff distance and Jaccard index.

A DISTANCE OF INTERVAL-VALUED BELIEF STRUCTURE

Definition 8 Given a frame of discernment Θ with n elements and its related interval-valued belief structures space S, a real function d: IBS×IBS→R^+ is called the metric distance between two interval mass functions m_1 and m_2 defined on Θ if d satisfies the following properties:

(d1) Nonnegativity: $d(m_1, m_2) \geq 0$; (d2) Nondegeneracy: $d(m_1, m_2) = 0$ □ $m_1 = m_2$; (d3) Symmetry: $d(m_1, m_2) = d(m_2, m_1)$; and (d4) Triangle inequality: $d(m_1, m_2) + d(m_1, m_3) \geq d(m_2, m_3)$, $\forall\, m_3$ of IBS.

Properties (d1) and (d2) together define positive definiteness.

We then propose to look for a distance measure between two IBSs m_1 and m_2 in the form

$$d\left(m_1, m_2\right) = \sqrt{\left(\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2\right)^{\mathrm{T}} \mathbf{S}\left(\vec{\mathbf{m}}_1 - \vec{\mathbf{m}}_2\right)}, \tag{21}$$

where $\left(\vec{\mathbf{m}}_1-\vec{\mathbf{m}}_2\right)$ is the distance measure between the intervals of A_i for m_1 and m_2. **S** is the similarity measure between the focal elements.

For any two subsets U and W of a Babach space Z the Hausdorff metric is simplified to de Carvalho and Pimentel (2012),

$$d_{\mathrm{H}}\left(U,W\right)=\max\left\{\sup_{u\in U}\inf_{w\in W}\left|u-w\right|,\sup_{w\in W}\inf_{u\in U}\left|u-w\right|\right\}. \tag{22}$$

If Z=R, U=[u_1, u_2], and W=[w_1, w_2] are intervals, using the L_1 norm (city block distance), the Hausdorff distance d_{H} between U and W is

$$d_{\mathrm{H}}\left(U,W\right)=\max\left(\left|u_1-w_1\right|,\left|u_2-w_2\right|\right). \tag{23}$$

The matrix **S** must be defined to describe the 'similarity' between the subsets of Θ. Jaccard index or Jaccard similarity coefficient defined in (11) was introduced by the botanist Paul Jaccard in 1901, and is now a classical and commonly used measure of similarity between sets in many applications. In this paper we also use the Jaccard index despite its ignorance of the frame of discernment Θ.

Definition 9 Let m_1 and m_2 be two IBSs defined on the same FOD Θ, containing n mutually exclusive and exhaustive hypotheses. The distance between m_1 and m_2 is defined by,

$$d_{\mathrm{IBS}}\left(m_1,m_2\right)=\sqrt{N_{\mathrm{f}}\cdot\mathbf{d}_{\mathbf{H}}^{\mathrm{T}}\cdot\mathbf{Jac}\cdot\mathbf{d}_{\mathbf{H}}}. \tag{24}$$

Here N_{f} is the normalization factor.

Jac is a $(2^n-1)\times(2^n-1)$ matrix whose elements are Jaccard indexes for all pairs of subsets A_i of Θ with n elements ($n=|\Theta|$), not including the empty set. The rows and columns of **Jac** are indexed by the elements A_i of $2^{\Theta}\backslash\emptyset$ and **Jac**(A_i, A_j) denotes the element of row A_i and column A_j of **Jac** in accordance with (11).

$\mathbf{d}_{\mathbf{H}}$ is a column vector of 2^n-1 dimensions, indexed by the sets in $2^{\Theta}\backslash\emptyset$ and $\mathbf{d}_{\mathbf{H}}^{\mathrm{T}}$ denotes the transpose vector of $\mathbf{d}_{\mathbf{H}}$. The vector $\mathbf{d}_{\mathbf{H}}$ is defined by

$$\mathbf{d}_{\mathbf{H}}\left(m_1,m_2\right)=\left[d_{\mathrm{H}}\left(m_1\left(A_1\right),m_2\left(A_1\right)\right),\ldots,d_{\mathrm{H}}\left(m_1\left(A_{2^n-1}\right),m_2\left(A_{2^n-1}\right)\right)\right]^{\mathrm{T}}. \tag{25}$$

Every element of $\mathbf{d}_{\mathbf{H}}$ is a Hausdorff distance between two interval-valued BBAs according to (23), and the ith element is

$$d_i=d_{\mathrm{H}}\left(m_1\left(A_i\right),m_2\left(A_i\right)\right),\ \text{for } i=1,2,\ldots,2^n\text{-}1, \tag{26}$$

where A_i is all subsets of Θ (excluding $\emptyset$).

PROPERTIES

The proposed distance measure between two IBSs has the following properties, for which we show the proofs.

Property 1. The normalizatio n factor $N_{\mathrm{f}}=\dfrac{n}{2\left(2n-1\right)}$.

We first recall some useful theorems for the proof.

Theorem 1 Given that **A** is $n\times n$ real symmetric positive definite, for any $n\times1$ column vector **x**, there exits the quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j\geq0$. If and only if for $\mathbf{x}=0$, $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}=0$.

Early in 1986, Gower (1971) proved that the Jaccard index matrix **Jac** is a positive semidefinite similarity matrix, and Bouchard et al. recently provided a proof for the positive definiteness. Here, we provide the conclusion as the following theorem.

Theorem 2 (Positive definiteness of **Jac**). The Jaccard index matrix formed by N arbitrary subsets of Θ a frame of reference of size n, is positive definite.

Proof of the normalization factor Suppose that $\Theta=\{\theta_1, \theta_2,\ldots,\theta_n\}$, m_1 and m_2 are two IBSs, and the Hausdorff distance between two elements A_i is $\mathbf{d}_{\mathrm{H}}=[d_1\ \ \mathrm{K}\ \ d_{2^n-1}]^{\mathrm{T}}$, according to (25) and (26).

Obviously, **Jac** is a real symmetric and positive definite matrix. By Theorem 1 and Theorem 2, **Jac** and $\boldsymbol{d}_{\mathbf{H}}$ can be translated into the quadratic form,

$$\begin{aligned}d&=\mathbf{d}_{\mathbf{H}}^{\mathrm{T}}\cdot\mathbf{Jac}\cdot\mathbf{d}_{\mathbf{H}}=\sum_{i=1}^{2^n-1}\sum_{j=1}^{2^n-1}d_id_j\frac{\left|A_i\cap A_j\right|}{\left|A_i\cup A_j\right|}=\sum_{i=1}^{2^n-1}d_i^2+2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1}d_id_j\frac{\left|A_i\cap A_j\right|}{\left|A_i\cup A_j\right|}\\&=d_1^2+\cdots+d_{2^n-1}^2+\cdots+\frac{2\left(n-1\right)}{n}\underbrace{\left(d_{2^n-n}d_{2^n-1}+\cdots+d_{2^n-2}d_{2^n-1}\right)}_{n}\end{aligned}. \tag{27}$$

Hence, can be rewritten as,

$$d_{\mathrm{IBS}}\left(m_1,m_2\right)=\sqrt{N_{\mathrm{f}}\cdot d}=\sqrt{N_{\mathrm{f}}\left(\sum_{i=1}^{2^n-1}d_i^2+2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1}d_id_j\frac{\left|A_i\cap A_j\right|}{\left|A_i\cup A_j\right|}\right)}. \tag{28}$$

For the sake of the validity and normalization of the interval-valued belief function, the maximum of distance value is reached when $d_{2^n-1}=1$, only one of the distances d_{2^n-1}, $d_{2^n-(n-1)}$, …, d_{2^n-2} equals 1, and the others are 0. Therefore, the maximum distance is $d=1+1+\dfrac{2\left(n-1\right)}{n}=\dfrac{2\left(2n-1\right)}{n}$, and the normalization factor is $N_{\mathrm{f}}=\dfrac{n}{2\left(2n-1\right)}$.

For example, when n=3, then Θ= {θ_1, θ_2, θ_3} and

$$\mathbf{Jac}^{(3)}=\begin{matrix} & \{\theta_1\}\{\theta_2\}\{\theta_3\}\{\theta_1\theta_2\}\{\theta_1\theta_3\}\{\theta_2\theta_3\}\{\theta_1\theta_2\theta_3\} \\ \begin{matrix}\{\theta_1\}\\\{\theta_2\}\\\{\theta_3\}\\\{\theta_1\theta_2\}\\\{\theta_1\theta_3\}\\\{\theta_2\theta_3\}\\\{\theta_1\theta_2\theta_3\}\end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/2 & 0 & 1/3 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 & 1/3 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 & 1/3 \\ 1/2 & 1/2 & 0 & 1 & 1/3 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 & 1/3 & 1 & 1/3 & 2/3 \\ 0 & 1/2 & 1/2 & 1/3 & 1/3 & 1 & 2/3 \\ 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 1 \end{bmatrix}\end{matrix}.$$

Let

$$\begin{aligned} d = \mathbf{d}_{\mathbf{H}}^{\mathrm{T}} \cdot \mathbf{Jac}^{(3)} \cdot \mathbf{d}_{\mathbf{H}} &= \left(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2 + d_7^2\right) \\ &+ \left(d_1 d_4 + d_1 d_5 + d_2 d_4 + d_2 d_6 + d_3 d_5 + d_3 d_6\right) \\ &+ \frac{2}{3}\left(d_1 d_7 + d_2 d_7 + d_3 d_7 + d_4 d_5 + d_4 d_6 + d_5 d_6\right) \\ &+ \frac{4}{3}\left(d_4 d_7 + d_5 d_7 + d_6 d_7\right) \end{aligned} . \quad (29)$$

The maximum distance is reached when d_7=1 and (d_4=1 or d_5=1 or d_6=1), and the others are 0, therefore, the maximum distance is d =3/10 and the normalization factor is N_f=1/ d=10/3.

Property 2 The distance d_{IBS} is a metric distance.

Proof: Nonnegativity. The Hausdorff distance is in the form of an absolute value, and the Jaccard index matrix is positive definite, therefore, the distance obviously d_{IBS} $(m_1, m_2) \geq 0$.

Nondegeneracy. Given two interval BBAs with mass distributions $\vec{m}_1 = \vec{m}_2$, the vector $\vec{m}_1 - \vec{m}_2 = 0$, and thus $\mathrm{d}_{\mathrm{IBS}}(m_1,m_2)$=0. Conversely, suppose $\mathrm{d}_{\mathrm{IBS}}(m_1,m_2)$=0. Since **Jac** is positive definite and is not a null matrix, it must be true that $\vec{m}_1 - \vec{m}_2 = 0$, i.e., $\vec{m}_1 = \vec{m}_2$.

Symmetry. Because the Hausdorff distance is symmetric, we can obtain $\mathbf{d_H}(m_1,m_2)$=$\mathbf{d_H}(m_2,m_1)$, and then $\mathrm{d}_{\mathrm{IBS}}$ (m_1, m_2)= $\mathrm{d}_{\mathrm{IBS}}$ (m_2, m_1).

Triangle inequality. Suppose that m_1, m_2 and m_3 are three IBSs defined on the same FOD Θ with size of n. Since the Hausdorff distance d_{H} defined in is a metric distance, the triangle inequality is naturally satisfied. That means for each A_i (i=1, ... , 2^n-1), there exists,

$$d_{\mathrm{H}}(m_1(A_i), m_2(A_i)) + d_{\mathrm{H}}(m_1(A_i), m_3(A_i)) \geq d_{\mathrm{H}}\,(m_2(A_i), m_3(A_i)). \quad (29)$$

Denote that $x_i = d_{\mathrm{H}}(m_1(A_i), m_2(A_i))$, $y_i = d_{\mathrm{H}}(m_1(A_i), m_3(A_i))$, and $z_i = d_{\mathrm{H}}(m_2(A_i), m_3(A_i))$, therefore,

$$x_i + y_i \geq z_i \Rightarrow \left(x_i + y_i\right)^2 \geq z_i^2 \Rightarrow \sum_{i=1}^{2^n-1}\left(x_i + y_i\right)^2 \geq \sum_{i=1}^{2^n-1} z_i^2. \quad (30)$$

According to the Mikowski inequality,

$$\sqrt{\sum_{i=1}^{2^n-1} x_i^2} + \sqrt{\sum_{i=1}^{2^n-1} y_i^2} \geq \sqrt{\sum_{i=1}^{2^n-1}\left(x_i + y_i\right)^2}, \quad (31)$$

there exists $\sqrt{\sum_{i=1}^{2^n-1} x_i^2} + \sqrt{\sum_{i=1}^{2^n-1} y_i^2} \geq \sqrt{\sum_{i=1}^{2^n-1} z_i^2}$. (32)

Square both sides of inequality to obtain

$$\sum_{i=1}^{2^n-1} x_i^2 + \sum_{i=1}^{2^n-1} y_i^2 + 2\sqrt{\sum_{i=1}^{2^n-1} x_i^2}\sqrt{\sum_{i=1}^{2^n-1} y_i^2} \geq \sum_{i=1}^{2^n-1} z_i^2. \quad (33)$$

Then, add some items on both sides of the inequality (33). Comparing this with the left side of (30), we have

$$\begin{aligned} &\sum_{i=1}^{2^n-1} x_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} x_i x_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|} + \sum_{i=1}^{2^n-1} y_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} y_i y_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|} \\ &+2\sqrt{\sum_{i=1}^{2^n-1} x_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} x_i x_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|}}\sqrt{\sum_{i=1}^{2^n-1} y_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} y_i y_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|}} \\ &\geq \sum_{i=1}^{2^n-1} z_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} x_i x_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|} + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} y_i y_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|} \\ &\geq \sum_{i=1}^{2^n-1} z_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} z_i z_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|} \end{aligned} \quad (34)$$

Hence,

$$\begin{aligned} &\left(\sqrt{N_f\left(\sum_{i=1}^{2^n-1} x_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} x_i x_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|}\right)} + \sqrt{N_f\left(\sum_{i=1}^{2^n-1} y_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} y_i y_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|}\right)}\right)^2 \\ &\geq \left(\sqrt{N_f\left(\sum_{i=1}^{2^n-1} z_i^2 + 2\sum_{i=1}^{2^n-1}\sum_{\substack{j=1\\j>i}}^{2^n-1} z_i z_j \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|}\right)}\right)^2 \end{aligned} \quad (35)$$

Then we obtain

$$d_{\mathrm{IBS}}\left(m_1, m_2\right) + d_{\mathrm{IBS}}\left(m_1, m_3\right) \geq d_{\mathrm{IBS}}\left(m_2, m_3\right). \quad (36)$$

Therefore, the property of triangle inequality of d_{IBS} is satisfied.

Through the mentioned analysis, it can be proved that conditions (d1) to (d4) are satisfied, i.e., the distance d_{IBS} is a metric distance.

Numerical Examples

EXAMPLES FOR THE CASE OF CRISP BELIEF STRUCTURES

Example 1 This example was proposed in Han et al. (2011) and reused in Han et al. (2014). In this example, m_1, m_2,...,

m_7 are seven crisp belief structures defined on the same FOD $\Theta=\{\theta_1, \theta_2, \theta_3\}$, as shown in Table 1.

The referenced BBA m_1 has a relatively large mass assignment value for the focal element $\{\theta_2\}$. Intuitively, for m_i (i=2,…,7), the larger the mass assignment value of $\{\theta_2\}$, the smaller the relative distance value. Furthermore, the minimum distance occurs at i=3, because m_3 has the maximum similarity to m_1. Seven different distances, d_J, d_T, d_C, d_{BI}, d_{Haus}, d_G and the proposed distance d_{IBS} in this paper, are employed. As shown in Figure 1, most of the change trends of the seven curves obtained by these distances are identical except for d_{Haus}. The result demonstrates that the proposed distance is effective for crisp belief structures.

Example 2 This example was proposed in Han et al. (2011) and reused in Han et al. (2014). Let $\Theta=\{\theta_1, \theta_2,\ldots, \theta_n\}$ be a fame of discernment. Three BBAs are defined as follows:

$$m_1(\theta_1) = m_1(\theta_2) = \ldots = m_1(\theta_n) = 1/n;$$

$$m_2(\Theta) = 1;\ m_3(\theta_k) = 1, \text{ for some } k\in\{1, 2,\ldots, n\}.$$

For a given number n, it is clear that m_1 is a Bayesian BBA, m_2 is a vacuous BBA and m_3 is a categorical BBA. The results of this experiment are displayed in Figure 2.

It can be noted that, $d_J(m_1,m_2)$ and $d_J(m_1,m_3)$ are superimposed for Jousselme distance d_J. This means that the Bayesian BBA m_1 is equidistant to the vacuous BBA m_2 and the categorical BBA m_3, because. It can also be seen that $d_T(m_1,m_2)$=0 because m_1 and m_2 have the same pignistic probabilities instead of BBAs. The conflict coefficient cannot be used as the distance metric since $d_C(m_1,m_2)= d_C(m_2,m_3)$=0. These examples illustrate that d_J, d_T and d_C are poor at discriminating the difference of the three BBAs. By contrast, the curves of d_{BI}, d_{Haus}, d_G and d_{IBS} are not superimposed. For a fixed number n, m_1 and m_3 are probability distributions which insist one focal element, as well as m_2 is ambiguous. Therefore, there is the reason to believe that m_1 is closer to m_3 than to m_2. However, d_{BI} accounts for the fact that $d_{BI}(m_1,m_3)$ is bigger than $d_{BI}(m_1,m_2)$, which is unreasonable. It is difficult how to select the tuning parameter K and α for the distances d_{Haus} and d_G. By comparison, the distance d_{IBS} show the better performance.

EXAMPLE FOR THE CASE OF INTERVAL-BASED BELIEF STRUCTURES

Example 3 Let $\Theta=\{\theta_1, \theta_2, \theta_3\}$ be a frame of discernment. The IVBs are all valid and normalized, and are defined in Table 2.

Intuitively, it can be seen from Table 2 that m_1, m_2 and m_3 give the biggest interval belief values to support θ_1, θ_2 and θ_3, respectively. In contrast, m_4 has the same interval values on θ_1, θ_2 and θ_3. m_5 supports the uncertain situation. The results of this experiment are displayed in Figure 3.

TABLE 1. BBAs of Example 1

Focal element	θ_1	θ_2	θ_3	$\theta_1\cup\theta_2$	$\theta_2\cup\theta_3$	$\theta_1\cup\theta_3$	$\theta_1\cup\theta_2\cup\theta_3$
m_1	0.1	0.8	0.1	0	0	0	0
m_2	0.8	0	0	0	0	0	0.2
m_3	0	0.8	0	0	0	0	0.2
m_4	0	0	0.8	0	0	0	0.2
m_5	0	0	0	0.8	0	0	0.2
m_6	0	0	0	0	0.8	0	0.2
m_7	0	0	0	0	0	0.8	0.2

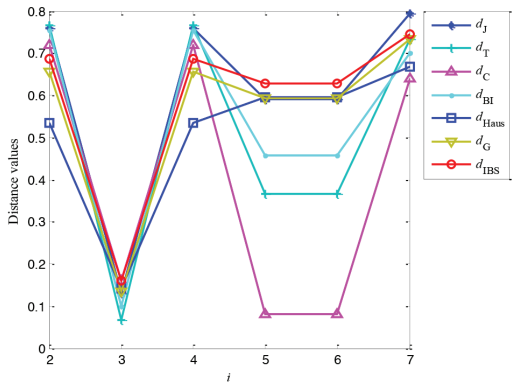

FIGURE 1. Distances between m_1 and m_i, i=2,…,7

TABLE 2. IVBs of Example 3

Focal element	θ1	θ_2	θ_3	$\theta_1 \cup \theta_2 \cup \theta_3$
m_1	[0.75, 0.8]	[0.05, 0.1]	[0.05, 0.1]	[0.1, 0.15]
m_2	[0.05, 0.1]	[0.75, 0.8]	[0.05, 0.1]	[0.1, 0.15]
m_3	[0.05, 0.1]	[0.05, 0.1]	[0.75, 0.8]	[0.1, 0.15]
m_4	[0.3, 0.4]	[0.3, 0.4]	[0.3, 0.4]	[0, 0]
m_5	[0.1, 0.2]	[0, 0]	[0, 0]	[0.8, 0.9]

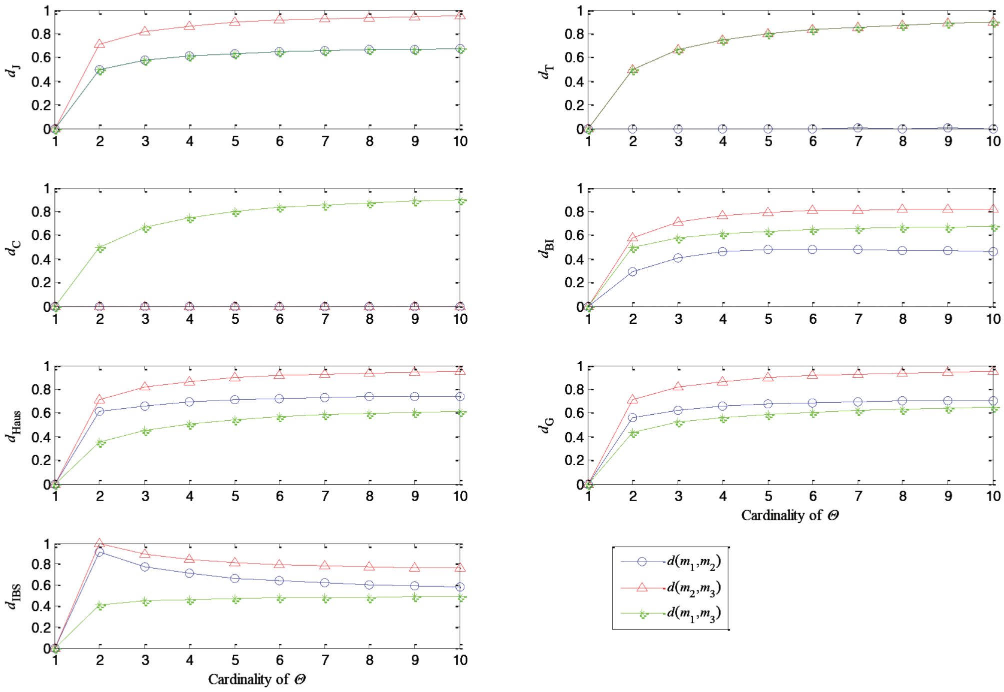

FIGURE 2. Distances between m_1, m_2 and m_3

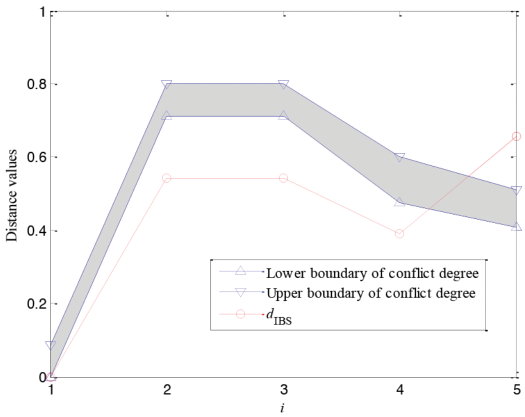

FIGURE 3. Distances between m_1 and m_i, i=1,…,5

In Figure 3, the shadow region is the range of conflict degree which can indirectly reflect the distance values between IVBs. When the conflict degree is bigger, it can be considered that the distance value is bigger, and vice versa. From 1 to 4, the curves are going in the same direction. It is equidistant at i=2 and 3. At i=4, m_4 has relative larger value on θ_1, therefore, the distance becomes closer. Meanwhile, at i=5 the conflict degree becomes smaller, which is unreasonable obviously. By contrast, the distance d_{IBS} also show the better performance.

CONCLUSION

In this paper, we have investigated the distance measurement problem in interval-valued belief structures. The proposed distance maintains the quadratic form structure using the Jaccard similarity index to compare the focal elements and the Hausdorff distance to measure the distance of interval numbers. Theoretical studies have shown it to be a full metric distance that satisfies the properties of non-negativity, non-degeneracy, symmetry and triangle inequality. Numerical experimental results showed that the proposed distance can be used in both crisp and interval-valued belief structures. None of the distance measures can be said to be superior to the others in the absolute and the choice of such a measure should always be guided by practical considerations relative to a specific application. Future work will include the following: Further detailing the formal properties of the surveyed measures; using real data for experimental comparisons; and using the distance in applications, such as conflict evidence combination, uncertainty measures, and belief function approximation.

ACKNOWLEDGEMENTS

This work was supported by the Major Science and Technology Program of Shanxi Province, China (Grant no. 20121101004). The authors declare that they have no competing interests.

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Received: 21 February 2019
Accepted: 23 December 2019